ON β-PASCU HARMONIC CONVEX FUNCTIONS OF ORDER α

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Abstract

We introduce a new class of β-pascu convex harmonic multivalent functions of order α in the open unit disc. We give sufficient conditions for these classes. These coefficient conditions are also shown to be necessary if the coefficients of analytic part of the harmonic functions are negative and the coefficients of co-analytic part of the harmonic function are positive. Among the results presented in this paper including extreme points, distortion bounds, convex combination and apply to integral operator for these classes are discussed.

Keywords: Harmonic function, β- Pascu Convex function

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1. Introduction

Let u, v be real harmonic functions in the simply connected domain Ω, then the continuous function $f=u+iv$ defined in Ω, is said to be harmonic in Ω. In any simply connected domain $\Omega\subset\mathbb{C}$ we can write $f=h+\overline{g}$, where h and g are analytic in Ω.

Let $H(p)$ be the family of functions $f=h+\overline{g}$ which are harmonic multivalent and sense-preserving in the open unit disk $U=\{z:|z|<1\}$. Let $H(p,t)$ for fixed positive integer p and t $\left(p,t\in Z^{+}\right)$ be the set of all harmonic multivalent and sense-preserving functions of the form $f(z)=h(z)+\overline{g(z)}$ with missing coefficients, where

$$h(z)=z^{p}+\sum_{n=p+t}^{\infty}a_{n}z^{n},\quad g(z)=\sum_{n=p+t-1}^{\infty}b_{n}z^{n}\ ,\quad \left|b_{p+t-1}\right|<1 \tag{1}$$

are analytic in U.

Let $HO(p,t)$ be the subclass of $H(p,t)$ consisting of functions of the form $f(z)=h(z)+\overline{g(z)}$ where h and g are given by

$$h(z)=z^p-\sum_{n=p+t}^{\infty}|a_n|z^n,\quad g(z)=\sum_{n=p+t-1}^{\infty}|b_n|z^n\ ,\quad |b_{p+t-1}|<1. \tag{2}$$

Definition 1 : A function $f\in H(p,t)$ is β-Pascu harmonic convex of order α if

$$\frac{1}{p}\operatorname{Re}\left\{\frac{(1-\beta)\left(zh'-\overline{zg'}\right)+\frac{\beta}{p}\left(zh'+\overline{zg'}+z^2h''+\overline{z^2g''}\right)}{(1-\beta)\left(h+\overline{g}\right)+\frac{\beta}{p}\left(zh'-\overline{zg'}\right)}\right\}>\alpha \tag{3}$$

where $p\in\mathbb{N}$, $\frac{p-1}{p}\le\alpha<1$, $\beta\ge 0$ and $z=re^{i\theta}$,

We denote by $H_{PC}(p,t,\beta,\alpha)$ the subclass of $H(p,t)$ consisting of β-Pascu harmonic convex of order α. If we take $g(z)\equiv 0$, we can obtain the class $TPC(p,m,\alpha,\beta)$ in [3] .We also let $HO_{PC}(p,t,\beta,\alpha)=H_{PC}(p,t,\beta,\alpha)\cap HO(p,t)$.

As β changes from 0 to 1, the family $H_{PC}(p,t,\beta,\alpha)$ produces a passage from the class of harmonic functions $HS^*(p,t,\alpha)\equiv H_{PC}(p,t,0,\alpha)$ consisting of function f where

$$\frac{1}{p}\operatorname{Re}\left\{\frac{zh'-\overline{zg'}}{h+\overline{g}}\right\}>\alpha \tag{4}$$

to the class of harmonic functions $HK(p,t,\alpha)\equiv H_{PC}(p,t,1,\alpha)$ consisting of function f where

$$\frac{1}{p}\operatorname{Re}\left\{\frac{z^2h''+zh'+\overline{z^2g''}+\overline{zg'}}{zh'-\overline{zg'}}\right\}>\alpha . \tag{5}$$

2. Coefficient Bounds

In our first theorem, we give a sufficient coefficient bound for harmonic functions in $H_{PC}(p,t,\beta,\alpha)$.

Theorem 1. Let $f=h+\overline{g}$ with h and g are given by (1). Then $f(z)\in H_{PC}(p,t,\beta,\alpha)$ if

$$\sum_{n=p+t}^{\infty}(n-p\alpha)\left[(1-\beta)p+n\beta\right]|a_n|+\sum_{n=p+t-1}^{\infty}(n+p\alpha)\left|(1-\beta)p-n\beta\right||b_n|\le p^2(1-\alpha) \tag{6}$$

where $p\in\mathbb{N}$, $\frac{p-1}{p}\le\alpha<1$, $\beta\ge 0$ and $z=re^{i\theta}$.

Proof: Using the fact that $\operatorname{Re} w \geq 0$ if and only if $|1+w| \geq |1-w|$ or and letting

$$w(z)=\frac{(1-\beta)\left[zh'-\overline{zg'}\right]+\frac{\beta}{p}\left[zh'+\overline{zg'}+z^2h''+\overline{z^2g''}\right]-p\alpha(1-\beta)\left[h+\overline{g}\right]-\alpha\beta\left[zh'-\overline{zg'}\right]}{(1-\beta)\left[h+\overline{g}\right]+\frac{\beta}{p}\left[zh'-\overline{zg'}\right]}$$

It is enough to show that $|1+w(z)|-|1-w(z)|\geq 0$.

Now, we have

$$|1+w(z)|=\left|\left[p^2(1-\alpha)+p\right]z^p+\sum_{n=p+t}^{\infty}(n-p\alpha)\left[p(1-\beta)+n\beta\right]+\left[p(1-\beta)+n\beta\right] - \sum_{n=p+t-1}^{\infty}(n+p\alpha)\left[p(1-\beta)-n\beta\right]-\left[p(1-\beta)-n\beta\right]b_n z^n\right|$$

$$=\left|\left[p^2(1-\alpha)+p\right]z^p+\sum_{n=p+t}^{\infty}(n-p\alpha+1)\left[p(1-\beta)+n\beta\right]a_n z^n-\sum_{n=p+t-1}^{\infty}(n+p\alpha-1)\left[p(1-\beta)-n\beta\right]b_n z^n\right|$$

$$|1-w(z)|=\left|\left[p^2(1-\alpha)-p\right]z^p+\sum_{n=p+t}^{\infty}(n-p\alpha)\left[p(1-\beta)+n\beta\right]-\left[p(1-\beta)+n\beta\right]a_n z^n - \sum_{n=p+t-1}^{\infty}(n+p\alpha)\left[p(1-\beta)-n\beta\right]+\left[p(1-\beta)-n\beta\right]b_n z^n\right|$$

$$=\left|\left[p^2(1-\alpha)-p\right]z^p+\sum_{n=p+t}^{\infty}\left[n-p\alpha-1\right]\left[p(1-\beta)+n\beta\right]a_n z^n-\sum_{n=p+t-1}^{\infty}\left[n+p\alpha+1\right]\left[p(1-\beta)-n\beta\right]b_n z^n\right|.$$

So by using (6) we have

$$|1+w(z)|-|1-w(z)|\geq 0$$

$$\geq\left[p^2(1-\alpha)+p\right]-\sum_{n=p+t}^{\infty}(n-p\alpha+1)\left[p(1-\beta)+n\beta\right]|a_n|-\sum_{n=p+t-1}^{\infty}(n+p\alpha-1)\left|p(1-\beta)-n\beta\right||b_n|$$

$$-\left[p-p^2(1-\alpha)\right]-\sum_{n=p+t}^{\infty}(n-p\alpha-1)\left[p(1-\beta)+n\beta\right]|a_n|-\sum_{n=p+t-1}^{\infty}(n+p\alpha+1)\left|p(1-\beta)-n\beta\right||b_n|$$

$$=2p^2(1-\alpha)-2\sum_{n=p+t}^{\infty}(n-p\alpha)\left[p(1-\beta)+n\beta\right]|a_n|-2\sum_{n=p+t-1}^{\infty}(n+p\alpha)\left|p(1-\beta)-n\beta\right||b_n|\geq 0$$

$$\sum_{n=p+t}^{\infty}(n-p\alpha)\left[p(1-\beta)+n\beta\right]|a_n|-\sum_{n=p+t-1}^{\infty}(n+p\alpha)\left|p(1-\beta)-n\beta\right||b_n|\leq p^2(1-\alpha)$$

and this completes the proof.

Remark 1: The coefficient bound (6) in previous theorem is sharp for the function

$$f(z)=z^{p}+\sum_{n=p+t}^{\infty}\frac{u_n}{(n-p\alpha)\left[(1-\beta)p+n\beta\right]}z^{n}+\sum_{n=p+t}^{\infty}\frac{\overline{v_n}}{(n+p\alpha)\left|(1-\beta)p-n\beta\right|}(\overline{z})^{n} \qquad (7)$$

where

$$\frac{1}{p^{2}(1-\alpha)}\left(\sum_{n=p+t}^{\infty}|u_n|+\sum_{n=p+t-1}^{\infty}|v_n|\right)=1.$$

We next show that the condition (6) is also necessary for functions in $HO_{PC}(p,t,\beta,\alpha)$.

Theorem 2 : Let $f=h+\overline{g}\in HO(p,t)$. Then $f(z)\in HO_{PC}(p,t,\beta,\alpha)$ if and only if

$$\sum_{n=p+t}^{\infty}(n-p\alpha)\left[(1-\beta)p+n\beta\right]|a_n|+\sum_{n=p+t-1}^{\infty}(n+p\alpha)\left|(1-\beta)p-n\beta\right||b_n|\le p^{2}(1-\alpha). \qquad (8)$$

Proof : From Theorem 1 and since $HO_{PC}(p,t,\beta,\alpha)\subset H_{PC}(p,t,\beta,\alpha)$ we conclude the “if” part. For the “only if part”, assume that $f(z)\in HO_{PC}(p,t,\beta,\alpha)$. Therefore, for $z=re^{i\theta}$, we have

$$=\operatorname{Re}\left\{\frac{p(1-\beta)\left[zh'(z)-z\overline{g'(z)}\right]+\beta\left(zh'(z)+\overline{zg(z)}+z^{2}h''(z)+\overline{z^{2}g''(z)}\right)}{p(1-\beta)\left[h(z)+\overline{g(z)}\right]+\beta\left[zh'(z)-z\overline{g'(z)}\right]}-p\alpha\right\}$$

$$=\operatorname{Re}\left\{\frac{z^{p}\left[p^{2}(1-\alpha)\right]-\sum_{n=p+t}^{\infty}(n-p\alpha)\left[(1-\beta)p+n\beta\right]|a_n|z^{n}-\sum_{n=p+t-1}^{\infty}(n+p\alpha)\left|(1-\beta)p-n\beta\right||b_n|\overline{z}^{n}}{pz^{p}-\sum_{n=p+t}^{\infty}\left[(1-\beta)p+n\beta\right]|a_n|z^{n}-\sum_{n=p+t-1}^{\infty}\left|n\beta-(1-\beta)p\right||b_n|^{n}\overline{z}^{n}}\right\}.$$

$$\ge\operatorname{Re}\left\{\frac{r^{p}\left[p^{2}(1-\alpha)\right]-\sum_{n=p+t}^{\infty}(n-p\alpha)\left[(1-\beta)p+n\beta\right]|a_n|r^{n}-\sum_{n=p+t-1}^{\infty}(n+p\alpha)\left|(1-\beta)p-n\beta\right||b_n|r^{n}}{pr^{p}-\sum_{n=p+t}^{\infty}\left[(1-\beta)p+n\beta\right]|a_n|r^{n}-\sum_{n=p+t-1}^{\infty}\left|n\beta-(1-\beta)p\right||b_n|r^{n}}\right\}$$

Letting $r\to 1^{-}$ we obtain

$$\frac{p^{2}(1-\alpha)-\left\{\sum_{n=p+t}^{\infty}(n-p\alpha)\left[(1-\beta)p+n\beta\right]|a_n|+\sum_{n=p+t-1}^{\infty}(n+p\alpha)\left|(1-\beta)p-n\beta\right||b_n|\right\}}{p-\sum_{n=p+t}^{\infty}\left[(1-\beta)p+n\beta\right]|a_n|-\sum_{n=p+t-1}^{\infty}\left|n\beta-(1-\beta)p\right||b_n|}\ge 0.$$

The above inequality holds for all $z\in U$. So if $z=r\to 1$ we obtain the required result (8). Now the proof of Theorem 2 is complete.

As special cases of Theorem 2, we obtain the following two correllaries.

Corrollary 1 : $f = h + \overline{g} \in \overline{HS^*}(p,t,\alpha) \equiv HS^*(p,t,\alpha) \cap HO(p,t)$ if and only if

$$\sum_{n=p+t}^{\infty} \frac{n - p\alpha}{p(1-\alpha)}|a_n| + \sum_{n=p+t-1}^{\infty} \frac{n + p\alpha}{p(1-\alpha)}|b_n| \leq 1.$$

Corrollary 2 : $f = h + \overline{g} \in \overline{HK}(p,t,\alpha) \equiv HK(p,t,\alpha) \cap HO(p,t)$ if and only if

$$\sum_{n=p+t}^{\infty} \frac{n(n - p\alpha)}{p^2(1-\alpha)}|a_n| + \sum_{n=p+t-1}^{\infty} \frac{n(n + p\alpha)}{p^2(1-\alpha)}|b_n| \leq 1.$$

3. Extreme Points

In the following theorem, we introduce extreme points of $HO_{PC}(p,t,\beta,\alpha)$.

Theorem 3 : $f = h + \overline{g} \in HO_{PC}(p,t,\beta,\alpha)$ if and only if it can be expressed as

$$f(z) = X_p z^p + \sum_{n=p+t}^{\infty} X_n h_n(z) + \sum_{n=p+t-1}^{\infty} Y_n g_n(z), \quad z \in U \tag{9}$$

where

$$h_n(z) = z^p - \frac{p^2(1-\alpha)}{(n - p\alpha)\left[(1-\beta)p + n\beta\right]} z^n \qquad (n = p+t, p+t-1, ...) \tag{10}$$

$$g_n(z) = z^p + \frac{p^2(1-\alpha)}{(n + p\alpha)\left|(1-\beta)p - n\beta\right|} \overline{z}^n \qquad (n = p+t-1, p+t, ...) \tag{11}$$

$$X_p \geq 0,\ Y_{p+t-1} \geq 0,\ X_p + \sum_{n=p+t}^{\infty} X_n + \sum_{n=p+t-1}^{\infty} Y_n = 1, \quad X_n \geq 0,\ Y_n \geq 0, \text{ for } n = p+t, p+t+1, ...$$

Proof : If $f(z)$ be given by (9), then

$$f(z) = z^p - \sum_{n=p+t}^{\infty} \frac{p^2(1-\alpha)}{(n - p\alpha)\left[(1-\beta)p + n\beta\right]} X_n z^n + \sum_{n=p+t-1}^{\infty} \frac{p^2(1-\alpha)}{(n + p\alpha)\left|(1-\beta)p - n\beta\right|} Y_n \overline{z}^n$$

Since by (8), we have

$$\sum_{n=p+t}^{\infty} (n - p\alpha)\left[(1-\beta)p + n\beta\right] \frac{p^2(1-\alpha)}{(n - p\alpha)\left[(1-\beta)p + n\beta\right]} |X_n|$$
$$+ \sum_{n=p+t-1}^{\infty} (n + p\alpha)\left|(1-\beta)p - n\beta\right| \frac{p^2(1-\alpha)}{(n + p\alpha)\left|(1-\beta)p - n\beta\right|} |Y_n|$$

$$= p^2(1-\alpha)\left(\sum_{n=p+t}^{\infty} |X_n| + \sum_{n=p+t-1}^{\infty} |Y_n|\right) = p^2(1-\alpha)(1 - X_p) \leq p^2(1-\alpha).$$

So $f(z) \in HO_{PC}(p,t,\beta,\alpha)$. Conversely, assume $f(z) \in HO_{PC}(p,t,\beta,\alpha)$, by putting

$$X_p = 1 - \left(\sum_{n=p+t}^{\infty} X_n + \sum_{n=p+t-1}^{\infty} Y_n \right),$$

where

$$X_n = \frac{(n - p\alpha)\left[(1-\beta)p + n\beta\right]}{p^2(1-\alpha)}|a_n|, \qquad Y_n = \frac{(n + p\alpha)\left|(1-\beta)p - n\beta\right|}{p^2(1-\alpha)}|b_n|.$$

We obtain

$$\begin{aligned} f(z) &= z^p - \sum_{n=p+t}^{\infty} |a_n| z^n + \sum_{n=p+t-1}^{\infty} |b_n| \overline{z}^n \\ &= z^p - \sum_{n=p+t}^{\infty} \frac{p^2(1-\alpha)X_n}{(n-p\alpha)\left[(1-\beta)p+n\beta\right]} z^n + \sum_{n=p+t-1}^{\infty} \frac{p^2(1-\alpha)Y_n}{(n+p\alpha)\left|(1-\beta)p-n\beta\right|} \overline{z}^n \\ &= z^p - \sum_{n=p+t}^{\infty} \left(z^p - h_n(z)\right) X_n + \sum_{n=p+t-1}^{\infty} \left(z^p - g_n(z)\right) Y_n \\ &= \left[1 - \left(\sum_{n=p+t}^{\infty} X_n + \sum_{n=p+t-1}^{\infty} Y_n \right)\right] z^p + \sum_{n=p+t}^{\infty} h_n(z) X_n + \sum_{n=p+t-1}^{\infty} g_n(z) Y_n \\ &= X_p z^p + \sum_{n=p+t}^{\infty} X_n h_n(z) + \sum_{n=p+t-1}^{\infty} Y_n g_n(z) \end{aligned}$$

that is the required representation.

4. Convex Combination

Now we show $HO_{PC}(p,t,\beta,\alpha)$ is closed under convex combination.

Theorem 4 : If $f_j\ (j=1,2,...)$ belongs to $HO_{PC}(p,t,\beta,\alpha)$, then the function $Q(z) = \sum_{j=1}^{\infty} \sigma_j f_j(z)$ is also in $HO_{PC}(p,t,\beta,\alpha)$, where $f_j(z)$ is defined by

$$f_j(z) = z^p - \sum_{n=p+t}^{\infty} a_{n,j} z^n + \sum_{n=p+t-1}^{\infty} b_{n,j} \overline{z}^n \qquad (j=1,2,......,\ 0 \le \sigma_j < 1,\ \sum_{j=1}^{\infty} \sigma_j = 1). \tag{12}$$

Proof : Since $f_j(z) \in HO_{PC}(p,t,\beta,\alpha)$, by (8) we have

$$\sum_{n=p+t}^{\infty} (n-p\alpha)\left[(1-\beta)p+n\beta\right]\left|a_{n,j}\right| + \sum_{n=p+t-1}^{\infty} (n+p\alpha)\left|(1-\beta)p-n\beta\right|\left|b_{n,j}\right| \le p^2(1-\alpha), \ (j=1,2,...).$$

Also

$$Q(z) = \sum_{j=1}^{\infty} \sigma_j f_j(z) = z^p - \sum_{n=p+t}^{\infty} \left(\sum_{j=1}^{\infty} \sigma_j a_{n,j} \right) z^n + \sum_{n=p+t-1}^{\infty} \left(\sum_{j=1}^{\infty} \sigma_j b_{n,j} \right) \overline{z}^n.$$

Now according to Theorem 2 we have

$$\sum_{n=p+t}^{\infty}(n-p\alpha)\left[(1-\beta)p+n\beta\right]\left|\sum_{j=1}^{\infty}\sigma_j a_{n,j}\right|+\sum_{n=p+t-1}^{\infty}(n+p\alpha)\left|(1-\beta)p-n\beta\right|\left|\sum_{j=1}^{\infty}\sigma_j b_{n,j}\right|$$

$$=\sum_{j=1}^{\infty}\left\{\sum_{n=p+t}^{\infty}(n-p\alpha)\left[(1-\beta)p+n\beta\right]\left|a_{n,j}\right|+\sum_{n=p+t-1}^{\infty}(n+p\alpha)\left|(1-\beta)p-n\beta\right|\left|b_{n,j}\right|\right\}\sigma_j$$

$$\leq p^2(1-\alpha)\sum_{j=1}^{\infty}\sigma_j=p^2(1-\alpha).$$

Thus $Q(z)\in HO_{PC}(p,t,\beta,\alpha)$.

Remark 2: We note that $HO_{PC}(p,t,\beta,\alpha)$ is a convex set.

5. Distortion Bounds

In the next theorem, we obtain the distortion bounds for $f(z)\in HO_{PC}(p,t,\beta,\alpha)$.

Theorem 5 : If $f=h+\overline{g}\in HO_{PC}(p,t,\beta,\alpha)$, $|z|=r<1$, then

$$|f(z)|\geq\left(1-\left|b_{p+t-1}\right|r^{t-1}\right)r^p-\left(\frac{p^2(1-\alpha)}{(p(1-\alpha)+t)\left|p-(2p+t)\beta\right|}-\frac{\left[p(1+\alpha)+t-1\right]\left|p(1-2\beta)+\beta(1-t)\right|}{(p(1-\alpha)+t)\left|p-(2p+t)\beta\right|}\left|b_{p+t-1}\right|\right)r^{p+t} \tag{13}$$

and

$$|f(z)|\leq\left(1+\left|b_{p+t-1}\right|r^{t-1}\right)r^p+\left(\frac{p^2(1-\alpha)}{(p(1-\alpha)+t)\left|p-(2p+t)\beta\right|}-\frac{\left[p(1+\alpha)+t-1\right]\left|p(1-2\beta)+\beta(1-t)\right|}{(p(1-\alpha)+t)\left|p-(2p+t)\beta\right|}\left|b_{p+t-1}\right|\right)r^{p+t} \tag{14}$$

Proof : Assume $f(z)\in HO_{PC}(p,t,\beta,\alpha)$ then by (8) we have

$$|f(z)|=\left|z^p-\sum_{n=p+t}^{\infty}|a_n|z^n+\sum_{n=p+t-1}^{\infty}|b_n|\overline{z}^n\right|=\left|z^p+\left|b_{p+t-1}\right|\overline{z}^{p+t-1}-\sum_{n=p+t}^{\infty}\left(|a_n|z^n-|b_n|\overline{z}^n\right)\right|$$

$$\geq r^p-\left|b_{p+t-1}\right|r^{p+t-1}-\frac{p^2(1-\alpha)}{(p+t-p\alpha)\left[(1-\beta)p-(p+t)\beta\right]}\sum_{n=p+t}^{\infty}\frac{(p+t-p\alpha)\left|(1-\beta)p-(p+t)\beta\right|}{p^2(1-\alpha)}\left(|a_n|+|b_n|\right)$$

$$\geq r^p-\left|b_{p+t-1}\right|r^{p+t-1}-\frac{p^2(1-\alpha)}{(p(1-\alpha)+t)\left|p-(2p+t)\beta\right|}\sum_{n=p+t}^{\infty}\left[\frac{(n-p\alpha)\left[(1-\beta)p+n\beta\right]}{p^2(1-\alpha)}|a_n|+\frac{(n+p\alpha)\left|(1-\beta)p-n\beta\right|}{p^2(1-\alpha)}|b_n|\right]r^{p+t}$$

$$\geq r^p-\left|b_{p+t-1}\right|r^{p+t-1}-\frac{p^2(1-\alpha)}{(p(1-\alpha)+t)\left|p-(2p+t)\beta\right|}\left(1-\frac{\left[p(1+\alpha)+t-1\right]\left|p(1-2\beta)+\beta(1-t)\right|}{p^2(1-\alpha)}\left|b_{p+t-1}\right|\right)r^{p+t}$$

$$= r^p - |b_{p+t-1}| r^{p+t-1} - \left(\frac{p^2(1-\alpha)}{(p(1-\alpha)+t)|p-(2p+t)\beta|} - \frac{[p(1+\alpha)+t-1]|p(1-2\beta)+\beta(1-t)|}{(p(1-\alpha)+t)|p-(2p+t)\beta|} |b_{p+t-1}| \right) r^{p+t}$$

Relation (14) can be proved by using the similar statements. So the proof is complete.

Corollary 3 : If $f(z) \in \overline{HS^*}(p,t,\beta,\alpha)$, then

$$|f(z)| \geq (1-|b_{p+t-1}| r^{t-1}) r^p - \left(\frac{p(1-\alpha)}{p(1-\alpha)+t} - \frac{|p(1+\alpha)+t-1|}{p(1-\alpha)+t} |b_{p+t-1}| \right) r^{p+t}$$

and

$$|f(z)| \leq (1+|b_{p+t-1}| r^{t-1}) r^p + \left(\frac{p(1-\alpha)}{p(1-\alpha)+t} - \frac{[p(1+\alpha)+t-1]}{p(1-\alpha)+t} |b_{p+t-1}| \right) r^{p+t} .$$

Corollary 4 : If $f(z) \in \overline{HK}(p,t,\beta,\alpha)$, then

$$|f(z)| \geq (1-|b_{p+t-1}| r^{t-1}) r^p - \left(\frac{p^2(1-\alpha)}{(p(1-\alpha)+t)(t+p)} - \frac{[p(1+\alpha)+t-1]|1-(p+t)|}{(p(1-\alpha)+t)(t+p)} |b_{p+t-1}| \right) r^{p+t}$$

and

$$|f(z)| \leq (1+|b_{p+t-1}| r^{t-1}) r^p + \left(\frac{p^2(1-\alpha)}{(p(1-\alpha)+t)(t+p)} - \frac{[p(1+\alpha)+t-1]|1-(p+t)|}{(p(1-\alpha)+t)(t+p)} |b_{p+t-1}| \right) r^{p+t} .$$

6. Integral Operator

Definition 2 ([5]) : The Jung-Kim-Srivastava integral operator is defined by

$$J^\sigma k(z) = \frac{(p+1)^\sigma}{2\Gamma(\sigma)} \int_0^z \left(\log \frac{z}{t} \right)^{\sigma+1} k(t)dt, \qquad \sigma > 0 \tag{15}$$

If $k(z) = z^p + \sum_{n=p+t}^{\infty} c_n z^n$, then

$$J^\sigma k(z) = z^p + \sum_{n=p+t}^{\infty} \left(\frac{p+1}{k+1} \right)^\sigma c_n z^n \tag{16}$$

also J^σ is a linear operator.

Remark 3 : If $f(z) = h(z) + \overline{g(z)}$, where

$$h(z) = z^p - \sum_{n=p+t}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=p+t-1}^{\infty} |b_n| z^n , \quad |b_{p+t-1}| < 1,$$

then

$$J^\sigma f(z) = J^\sigma h(z) + \overline{J^\sigma g(z)}. \tag{17}$$

Theorem 6 : If $f(z) \in HO_{PC}(p,t,\beta,\alpha)$ and $p < k$, then $J^\sigma f(z)$ is also in $HO_{PC}(p,t,\beta,\alpha)$.

Proof : By (16) and (17), we obtain

$$J^{\sigma} f(z) = J^{\sigma}\left(z^p - \sum_{n=p+t}^{\infty} a_n z^n + \sum_{n=p+t-1}^{\infty} b_n \overline{z}^n \right)$$

$$= z^p - \sum_{n=p+t}^{\infty} \left(\frac{p+1}{k+1}\right)^{\sigma} a_n z^n + \sum_{n=p+t-1}^{\infty} \left(\frac{p+1}{k+1}\right)^{\sigma} b_n \overline{z}^n .$$

Since $f(z) \in HO_{PC}(p,t,\beta,\alpha)$, then by Theorem 2 we have

$$\sum_{n=p+t}^{\infty} (n - p\alpha)\left[(1-\beta)p + n\beta\right]|a_n| + \sum_{n=p+t-1}^{\infty} (n + p\alpha)\left|(1-\beta)p - n\beta\right||b_n| \le p^2(1-\alpha) \qquad (18)$$

we must show

$$\sum_{n=p+t}^{\infty} (n - p\alpha)\left[(1-\beta)p + n\beta\right]|a_n|\left(\frac{p+1}{k+1}\right)^{\sigma} + \sum_{n=p+t-1}^{\infty} (n + p\alpha)\left|(1-\beta)p - n\beta\right||b_n|\left(\frac{p+1}{k+1}\right)^{\sigma} \le p^2(1-\alpha) \qquad (19)$$

But in view of (18) the inequality in (19) holds true if $\left(\frac{p+1}{k+1}\right)^{\sigma} < 1$, since $\sigma > 0$, therefore (19) holds true if $p < k$ and this gives the result .

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