New conditions for the absolute stability of a certain lurie system

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Abstract

In this paper, we investigate the absolute stability of a certain Lurie system of the form (2.1) where A's are matrices and C and q are vectors having appropriate dimensions. The nonlinearities of (2.1) which are $fi, i = 1, 2..,$ are continuous, and they are our main focus of investigation in this study, the degenerate system that gives unique equilibrium state $(x^t, y^t)^t = 0$ help us to take the derivative of the nonlinearities of (2.1) which resulted to (2.2), and (2.3) described the boundary layer of (2.1).

The assumption $C_{ii}^t A_{ii}^{-1} q_i > 0$ holds, together with some notations that were introduced with subsystem (2.2) (2.3) enable us to introduce the Lyapunov matrix-valued function which is the main tool for this study, that enable us to prove the main results, (2.4) gives our matrix-valued function, and scalar functions were introduced on (2.4) that lead us to (2.5). we introduced estimates which satisfy the estimates of the matrix-valued functions that gives (2.6), one of the conditions for Lyapunov matrix-valued function to be stable is that the derivative of the given function must be negative – definite at the given interval, and the function must be positive – definite, this was shown under the statement of the main results, where we established sufficient conditions that guarantees the absolute stability of the equation of the form (2.1).

Keywords: Absolute Stability, Lurie System, Singularly Perturbed, Nonlinearities Liapunov Matrix-Valued function.

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1. Introduction

Singularly-perturbed systems are known to be rather widely used in the engineering and technology as models of real processes. (see e.g. surveys by Vasilieva and Butuzov[15]; Kokotovic O' Malley, and Sannuti [8]; Grujic[2, 3]; and some others). Stability properties were studied by Klimushev and Krasovskii[7,], Hoppensteadt [4,5] Siljak[14] Zien[18].

Impressive results have been obtained on the stability of control systems using frequency domain ideas over the year outstanding examples of such works can be found in the Articles of Kalman [6], Popov[12] and Yacobovich [17] arising in their quests to solve Lurie's problems[9] in automatic controls. More expository results can be found in [1, 10, 11,13, and 16]

2. Preliminaries

In this paper we consider the autonomous singularly perturbed system of Lurie type

$$\frac{dx}{dt} = A_{11}x + A_{12}y + q_1 f_1(\sigma_1), \quad \sigma_1 = C_{11}^T + C_{12}^T y;$$

$$\mu\frac{dy}{dt} = A_{21}x + A_{22}y + q_2 f_2(\sigma_2), \quad \sigma_2 = C_{21}^T + C_{22}^T y, \rightarrow 2.1$$

Where $x \epsilon N_x \subseteq R^n, y \epsilon N_y \subseteq R^m$ $\mu \epsilon (0,1]$ is a small parameter, the matrices $A(.)$ and the vectors $c(.), q(.)$ having appropriate dimensions. The nonlinearities $f_i, i = 1,2;$ are continuous, $f_i(0) = 0$ and in the Lurie sectors $[0,k_i]$, $k_i \epsilon (0,+\infty)$ satisfy the conditions $f_i(\sigma_i)/\sigma_i \epsilon$ $(0,k_i]$, $i = 1,2$; $\forall\ \sigma_i \epsilon\ (-\infty,+\infty)$.

In this paper, we study only those nonlinearities f_i for which the state $(x^T, y^T)^T = 0$ is the unique equilibrium state of the degenerate system.

$$\frac{dx}{dt} = A_{11}x + q_1 f_1(\sigma_1^0);\ \ \sigma_1^0 = C_{11}^T x \rightarrow 2.2$$

and of the system, describing the boundary layer,

$$\mu\frac{dy}{dt} = A_{22}y + q_2 f_2(\sigma_2^0);\ \ \sigma_2^0 = C_{22}^T y \rightarrow 2.3.$$

This assumption holds if $C_{ii}^T\ A_{ii}^{-1}\ q_i > 0$.

The following notations are introduced;

$f(x,0) = A_{11}x + q_1 f_1(\sigma_1^0)$;

$f^*(x,y) = A_{12}y + q_1[f_1(\sigma_1) - f_1(\sigma_1^0)]$;

$g(0,y) = A_{22}y + q_2 f_2(\sigma_2^0)$;

$g^*(x,y) = A_{21}x + q_2[f_2(\sigma_2) - f_2(\sigma_2^0)]$

Then the system (2.1) takes the form

$$\frac{dx}{dt} = f(x,0) + f^*(x,y);$$

$$\mu\frac{dy}{dt} = g(o,y) + g^*(x,y)$$

Together with system (2.1) and subsystems (2.2)(2.3) we shall consider the matrix-valued function

$$u(x,y,\mu) = \begin{pmatrix} v_{11}(x) & v_{12}(x,y,\mu) \\ v_{21}(x,y,\mu) & v_{22}(y,\mu \end{pmatrix};\ v_{12} = v_{21} \rightarrow (2.4)$$

where

$v_{11} = x^T B_1 x$; $v_{12} = \mu y^T B_2 y$; $v_{12} = \mu x^T B_3 y$; where $B_1 and\ B_2$ are symmetric, positive-definite matrices; B_3 is a constant matrix.

We introduces scalar function on (2.4) to obtained $v(x,y,\mu) = \eta^{Tu}(x,y,\mu)\boldsymbol{\eta} \to (2.5)$. where $\eta^T = (\eta_1\,\eta_2)$; $\boldsymbol{\eta}\,\boldsymbol{\epsilon R^2}+$; $\boldsymbol{\eta_i} > 0, i = 1,2$

we assume that the elements of the matrix-valued function (2.4) satisfy the following estimates

$v_{11}(x) \geq \lambda_m(B_1)\|x\|^2\ \ \forall\, x\epsilon\, N_{x_0} = \{x : x\epsilon\, N_x; x \neq 0\}$;

$v_{22}(y,\mu) \geq \mu\lambda_m(B_2)\|y\|^2\ \ \forall\,(y,\mu)\epsilon\, N_{y_0} \times m; \to (2.6)$

$v_{12}(x,y,\mu) \geq -\mu\lambda_m^{1/2}(B_3\, B_3^T)\|x\|\,\|y\|\ \forall\,(x,y,\mu)\epsilon\ \ N_{x_0} \times N_{y_0} \times, M$

where $\lambda_m(B_i)$ are the minimal eigenvalues of the matrices $B_i, i = 1,2$; $\lambda_m^{1/2}(B_3\, B_3^T)$ is the norm of the matrix $(B_3\, B_3^T)$; $\lambda_m(B_3\, B_3^T)$ is the maximal eigenvalue of the matrix $B_3\, B_3^T$; $N_{y_0} = \{y : y\epsilon\, N_y; y \neq 0\}$; $M = (0,1]$. The equation (2.5) have the estimate as follows

$v(x,y,\mu) \geq U^T H^T A H U\ v(x,y,\mu)\epsilon N_x \times N_y \times$M Where $U^T = (\|x\|\,\|y\|)$; $H = diag\,(\eta_1\,\eta_2)$;

$$A(\mu) = \begin{pmatrix} \lambda_m(B_1) & -\mu\lambda_m^{1/2}(B_3\, BT_3) \\ -\mu\lambda_m^{1/2}(B_3\, B_3^T) & \mu\lambda_m(B_2) \end{pmatrix}$$

For the derivatives of the elements of the matrix-valued function (2.4) along the solutions of the system (2.1) we have the following estimates

(a).$(\nabla x\, v_{11})^T f(x,0) \leq P_{11}\|x\|^2\ \forall\, x\epsilon N_{x_0}$;

(b).$(\nabla x\, v_{11})^T f^*(x,y) \leq P_{11}\|x\|^2 + 2P_{13}^{1/2}\|x\|\,\|y\|\ \ \ \forall\,(x,y,)\epsilon\ \ N_{x_0} \times N_{y_0}$;

(c).$(\nabla_y\, v_{22})\ T_g(0,y) \leq \mu P_{21}\|y\|^2\ \forall(y,m)\epsilon N_{y_0} \times M$

(d).$(\nabla_y\, v_{22}) T g^*(x,y) \leq \mu P_{22}\|y\|^2 + \mu P_{23}^{1/2}\|x\|\,\|y\|\ \ \ \forall\,(x,y,\mu)\epsilon\ \ N_{x_0} \times N_{y_0} \times M$

(e). $(\nabla_x\, v_{12}) T f(x,0) \leq \mu P_{15}^{1/2}\|x\|\,\|y\|\ \ \ \forall\,(x,y,\mu)\epsilon\ \ N_{x_0} \times N_{y_0} \times M$;

(f). $(\nabla_x\, v_{12}) T f^*(x,y) \leq \mu P_{17}^{1/2}\|x\|\,\|y\| + \mu P_{18}\|y\|^2\ \forall\,(x,y,\mu)\epsilon\ \ N_{x_0} \times N_{y_0} \times M \to (2.7)$

(g).$(\nabla y\ v_{12})^T g(0,y) \leq \mu P_{25}^{1/2} \|x\|\ \|y\|\ \ \forall\ (x,y,\mu)\epsilon\ \ N_{x_0} \times N_{y_0} \times M;$

(h).$(\nabla y\ v_{12}) T g^*(x,y) \leq \mu P_{26} \|x\|^2 + \mu P_{27}^{1/2} \|x\|\ \|y\|\ \epsilon N_x \times N_y \times M$

Where $P_{11}, P_{12}, P_{21},\ P_{22},\ P_{18}\ \ P_{26}$ are the maximal eigenvalues of the matrices

$B_1\ A_{11} + A_{11}^T\ B_1 + B_1 q_1 K_1^*\ C_{11}^T + (q_1\ K_1^*\ C_{11}^T)^T B_1,$

$B_2 A_{22} + A_{22}^T\ B_2 + B_2 q_2 K_2^* C_{22}^T + (q_2 K_2^* C_{22}^T)^T B_2,$

$B_2 q_2 K_2^* C_{22}^T + (q_2 K_2^* C_{22}^T)^T B_2,$

$A_{12}^T\ B_3 + (q_1 K_1^* C_{12}^T) T B_1$

$A_3\ A_{21} + B_3\ q_2\ K_2^*\ C_1^T$ Respectively;

Where $P_{13}^{1/2}, P_{23}^{1/2}, P_{15}^{1/2}, P_{17}^{1/2}, P_{25}^{1/2}\ P_{27}^{1/2}$ Are the norms of the matrices.

$B_1\ A_{12} + B_1 q_1 K_1^* C_{12}^T,$

$B_2\ A_{21} + B_2 q_2 K_2^* C_{21,}^T$

$A_{11}^T\ B_3 + (q_1 K_1^* C_{12}^T)^T B_3,$

$(q_1 K_1^* C_{12}^T)^T B_3,$

$B_3\ A_{22} + B_3 q_2 K_2^* C_{22,}^T$

$B_3 q_2 K_2^* C_{22,}^T$ Respectively.

$$K_i^* = \begin{Bmatrix} k_i\ for\ \sigma_i q_i\ B_j x > 0 \big(or\ \sigma_i q_i\ B_j y > 0\ \big); \\ 0\ for\ \sigma_i q_i\ B_j x \leq 0 \big(or\ \sigma_i q_i\ B_j y \leq 0\ \big)\ \begin{Bmatrix} i = 1,2 \\ j = 1,2,3 \end{Bmatrix} \end{Bmatrix}$$

Denoting the upper bound of the derivative of the function (2-5)

By $\frac{d}{dt} v_m(x,y,\mu)$, we find the estimate. $\frac{d}{dt} v_m(x,y,\mu) \leq u^T c\ (\mu) u, \rightarrow$ (2.8)

Where $c(\mu) = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}, \sigma_{12} = \sigma_{21};$

$\sigma_{11} = \eta_1^2(P_{11} + P_{12}) + 2\eta^1\, \eta_2\, P_{26};$

$\sigma_{22} = \eta_2^2(P_{21} + P_{22}) + 2\mu\, \eta_1\eta_2\, P_{18};$

$\sigma_{12} = \eta_1^2\, P_{13}^{1/2} + \eta_2^2 P_{23}^{1/2} + \eta_1\eta_2 \left(\mu P_{15}^{1/2} + \mu P_{17}^{1/2} + P_{25}^{1/2} + P_{27}^{1/2}\right)$

We introduce the quantities

$\mu_1 = \frac{-\eta_2(P_{21}+P_{22})}{2\eta_1 P_{18}};\ \mu_2 = \frac{-b\pm\sqrt{b^2-4ac}}{2a};\ \mu_0 = \min(\mu_1, \mu_2)$

Where $a = \eta_1^2\, \eta_2^2\, \left(P_{15}^{1/2} + P_{17}^{1/2}\right)^2;$

$$b = \eta_2\eta_2 \left(P_{15}^{1/2} + P_{17}^{1/2}\right)\left[\eta_1^2\, P_{13}^{1/2} + \eta_2^2 P_{23}^{1/2} + \eta_1\eta_2 \left(P_{25}^{1/2} + P_{27}^{1/2}\right)\right] - 2\, \eta_1\eta_2\, P_{18}\, \sigma_{11};$$

$$c = \left[\eta_1^2\, P_{13}^{1/2} + \eta_2^2 P_{23}^{1/2} + \eta_1\eta_2 \left(P_{25}^{1/2} + P_{27}^{1/2}\right)\right]^2 - \eta_2^2(P_{21} + P_{22})\sigma_{11}$$

Implies that $\mu_0 > 1$, then we consider $\mu\, \epsilon(0,1]$

3. Statement of the main results

Proposition 3.1. The matrix $c(\mu)$ is negative-definite for every $\mu\, \epsilon(0,1]$ and for $\mu \rightarrow 0$ if the following conditions hold;

(a).$\sigma_{11} < 0$

(b).$\eta_1 P_{18} > 0$

(c).$\eta_2(P_{21} + P_{22}) < 0$

(d).$c < 0$

Remark 3.1. If $\eta_1 P_{18} \leq 0$ and the conditions (a),(b),(d) of proposition 3.1 are satisfied, then its assertion remains valid for $\mu_0 = \mu_2$

Theorem 3.1. Assume that the singularly perturbed Lurie system (2.1) is such that the matrix-valued function (2.4) has been constructed for it, the elements of which satisfy the estimates (2.6) and for the upper bound of the derivative of the function (2.5) the estimate (2.7) holds in this case ,if

(a). The matrix A is positive-definite;

(b). The matrix $c(\mu)$ is negative-definite for every $\mu \epsilon(0, \mu_0)$ and for μ→ 0

Then the equilibrium state $(x^T, y^T) = 0$ of the system (2.1) is uniformly asymptotically stable for every $\mu \epsilon(0, \mu_0)$ and for μ→ 0 if, furthermore, $N_x \times N_y = R^{n+m}$ Then the equilibrium state of the system (2.1) is uniformly asymptotically stable on the whole for every $\mu \epsilon(0, \mu_0)$ and for μ→ 0.

Proof. On the basis of the matrix-valued function (2.4) with the aid of the vector $\eta \epsilon R^2 +, \boldsymbol{\eta} > \mathbf{0}$, we construct the scalar function (2.5) under the estimates (2.6) one can show that $v(x, y, \mu) \geq U^T H^T A H U$ $\forall (x, y, \mu) \epsilon N_x \times N_y$ ×M. Then from condition (a) of theorem 3.1 there follows that the function $v(x, y, \mu)$ is positive-definite. For the derivative $\frac{d}{dt} v(x, y, \mu)$ the estimate (2.7) holds from here and from condition (b) of the theorem 3.1 there follows that the derivative $\frac{d}{dt} v(x, y, \mu)$ of the function (2.5) is negative-definite for every $\mu \epsilon(0, \mu_0)$ and for μ→ 0. As is known (see Grujic, Martynyuk and Ribbens-Pavella [3]), these conditions are sufficient for the uniform asymptotic stability of the equilibrium state of the system (2.1). In this case $N_x \times N_y = R^{n+m}$ the function $v(x, y, \mu)$ is radially unbounded which, together with the other conditions, proves the second assertion of this theorem. This is the absolute stability of the system (2.1), μ_0 being an estimate of the upper bound of the variation of the parameter μ.

This complete the proof.

EXAMPLE: we consider a system of the form

$\mu \frac{dy}{dt} = \mu A_{21} x + A_{22} y + q_2 f_2(\sigma_2), \sigma_2 = C_{21}^T x + C_{22}^T y$ in which

$A_{11} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$ $q_1 \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}$; $C_{11} = \begin{pmatrix} -0.01 \\ 0 \end{pmatrix}$;

$A_{12} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $C_{12} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $(K_1 = 2)$;

$A_{21} = \begin{pmatrix} 0.001 & 0 \\ 0 & 0.001 \end{pmatrix}$; $C_{21} = \begin{pmatrix} 0.001 \\ 0 \end{pmatrix}$; $q_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$;

$A_{22} = \begin{pmatrix} -4 & 1 \\ 1 & 4 \end{pmatrix}$; $C_{22} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $(K_2 = 1)$;

The matrix-valued function (2.4) has the elements $v_{11} = x^T \begin{pmatrix} 0.3 & 0.1 \\ 0.1 & 0.3 \end{pmatrix} x$;

$v_{22}(y,\mu) = \mu y^T \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} y$;

$v_{12}(x,y,\mu) = v_{21}(x,y,\mu) = Mx^T \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix} y$,

For which we have the estimates

$v_{11}(x) \geq 0 \cdot 2\|x\|^2$; $v_{22}(y,\mu) \geq 2\mu\|y\|^2$;

$v_{12}(x,y,\mu) \geq -0.01\mu\|x\| \, \|y\|$

If $\eta_i = 1, i = 1,2$ then the matrix $A = \begin{pmatrix} 0.2 & -0.01\mu \\ -0.01\mu & 2\mu \end{pmatrix}$

which is positive-definite for every $\mu \epsilon \, (0,1)$.

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